Decay rates and cross sections

Fermi's Golden Rule (FGR) for a transition rate:

$$\Gamma_{fi} = 2\pi \, |T_{fi}|^2 \, \rho(E_i)$$

$|T_{fi}|$: transition matrix element (fundamental physics)

$\rho(E_i)$: density of states (phase space, kinematics)

$$T_{fi} = \langle f | \hat{H}' | i \rangle + \sum_{j \neq i} \frac{\langle f | \hat{H}' | j \rangle \langle j | \hat{H}' | i \rangle}{E_i - E_j} + \dots$$

$$\rho(E_i) = \left| \frac{dn}{dE} \right|_{E_i}$$ ← # accessible states between E and $E+dE$

Alternatively $$\left| \frac{dn}{dE} \right|_{E_i} = \int \frac{dn}{dE} \, \delta(E_i - E) \, dE$$

We use relativistic QM → FGR with density of states is based on relativistic treatment of phase space + relativistic normalisation

First order perturbation theory

$$T_{fi} = \langle \psi_1 \psi_2 | \hat{H}' | \psi_a \rangle = \int_V \psi_1^* \psi_2^* \hat{H}' \psi_a \, d^3\vec{x}$$

the integral extends over the volume of which the WF are normalised

Born approximation

$$\psi(\vec{x},t) = A e^{i(\vec{p}\cdot\vec{x} - Et)}$$

Non-relativistic density: $\rho = \psi^* \psi$

$$\Rightarrow \int_0^a \int_0^a \int_0^a \psi^* \psi \, dx\,dy\,dz = 1 \Rightarrow A^2 = \frac{1}{V}$$

Periodicity $\psi(x+a, y, z) = \psi(a, y, z)$

$$(p_x, p_y, p_z) = (n_x, n_y, n_z)\frac{2\pi}{a}$$

The condition implies standing waves of the form:

$$\psi(x,y,z) = A \sin(p_x x)\sin(p_y y)\sin(p_z z)$$

Since $\sin(p_x x) = \dfrac{e^{ip_x x} - e^{ip_x x}}{2i}$ → forward and backward propagating components

$$d^3\vec{p} = dp_x\, dp_y\, dp_z = \left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$

states dn with p between p and $p+dp$ (volume in a spherical shell a momentum p divided by the volume occupied by a single state: $\frac{(2\pi)^3}{V}$)

$$dn = 4\pi p^2 dp \cdot \frac{V}{(2\pi)^3}$$

$$\frac{dn}{dp} = \frac{4\pi p^2}{(2\pi)^3} \cdot V$$

$$\rho(E) = \frac{dn}{dE} = \frac{dn}{dp} \cdot \left|\frac{dp}{dE}\right|$$

density of states increases with the momentum of the final-state particle $\Rightarrow$ decays to lighter particles preferred (only by phase space)

Volume dependence on $\frac{dn}{dE}$ cancelled by WF normalisation

$\Rightarrow$ set $V = 1$

infinitesimal volume in p-space

$$\boxed{dn_i = \frac{d^3\vec{p}_i}{(2\pi)^3}}$$

$$dn = \prod_{i=1}^{N-1} dn_i = \prod_{i=1}^{N-1} \frac{d^3\vec{p}_i}{(2\pi)^3}$$

including the volume element of the final particle and enforcing momentum conservation:

$$\boxed{dn = (2\pi)^3 \prod_{i=1}^{N} \frac{d^3 p_i}{(2\pi)^3} \cdot \delta^3\left(p_a - \sum_{i=1}^{N} p_i\right)}$$

Lorentz-invariant phase space

- volume contracted by $1/\gamma$ depending on the ref. frame
$\Rightarrow$ 1 particle/unit volume $\sim \gamma = E/m$

LI volume $\Rightarrow$ E particles/unit volume in the boosted frame. Convention $2E$ instead of E

WF $\quad \int_V \psi^* \psi \, d^3\vec{x} = 1 \Rightarrow \int_V \psi'^* \psi' \, d^3\vec{x} = 2E$

$\psi' = \sqrt{2E}\,\psi \qquad M_{fi} = \langle \psi_1' \psi_2' \ldots | \hat{H}' | \psi_a' \psi_b' \ldots \rangle =$

$= \sqrt{2E_1 2E_2 \ldots 2E_a 2E_b \ldots}\; T_{fi}$

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_a - E_1 - E_2)\, dn$$

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a} \int |M_{fi}|^2 \delta(E_a - E_1 - E_2)\, \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\, \frac{d^3p_1}{(2\pi)^3 2E_1} \cdot \frac{d^3p_2}{(2\pi)^3 2E_2}$$

$\frac{d^3p_1}{(2\pi)^3 2E_1} \rightarrow$ LI phase space

Proove $d^3p' = dp_x' dp_y' dp_z' = dp_x dp_y \frac{dp_z'}{dp_z} dp_z = \frac{dp_z'}{dp_z} d^3p$

(Boost along z)

$$E^2 = p_x^2 + p_y^2 + p_z^2 + m^2 \qquad p_z' = \gamma(p_z - \beta E)$$

$$E' = \gamma(E - \beta p_z)$$

$$\frac{dp_z'}{dp_z} = \gamma\left(1 - \beta\frac{p_z}{E}\right) = \frac{\gamma}{E}(E - \beta p_z) = \frac{E'}{E}$$

$$\Rightarrow \boxed{\frac{d^3p'}{E'} = \frac{d^3p}{E}}$$

More general LI phase space

4-momentum

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a} \int (2\pi)^{-6} |M_{fi}|^2 \delta^4(p_a - p_1 - p_2)\, \delta(p_1^2 - m_1^2)\, \delta(p_2^2 - m_2^2)\, d^4p_1\, d^4p_2$$

by using $\frac{1}{2E_i} = \int \delta(E_i^2 - p_i^2 - m^2)\, dE_i$

L4 start here

Decay rate calc. $i \to 1+2$

Use CoM frame: $E_i = m$ $\vec{p}_i = 0$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\delta^3(\vec{p}_1 + \vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}$$

Integrate over $\vec{p}_2$ using δ-function $|\vec{p}_1| = |\vec{p}_2| = |\vec{p}^*|$, $\vec{p}_1 = -\vec{p}_2$

$$\Gamma_{fi} = \frac{1}{8\pi^2 m_a} \int |M_{fi}|^2 \frac{d^3\vec{p}_1}{4E_1E_2} \cdot \delta(m_i - E_1 - E_2)$$

$E_2^2 = m_2^2 + p_1^2$

in spherical coordinates: $d^3p_1 = p_1^2 dp_1 \sin\theta \, d\theta \, d\varphi =$
$= p_1^2 dp_1 d\Omega$

$$\Rightarrow \Gamma_{fi} = \frac{1}{8\pi^2 m_a} \int |M_{fi}|^2 \delta\left(m_a - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right) \frac{p_1^2}{4E_1E_2} dp_1 d\Omega$$

Use δ-function $\int g(p_1)\delta(f(p_1))\,dp_1$

$f(p_1)$: $m_a - E_1 - E_2$ $\qquad g(p_1) = \frac{p_1^2}{4E_1E_2}$

Use $\int |M_{fi}|^2 g(p_1)\,\delta(f(p_1))\,dp_1 = |M_{fi}|^2 g(p^*) \left|\frac{df}{dp_1}\right|_{p^*}^{-1}$

$$\frac{df}{dp_1} = \frac{p_1}{\sqrt{m_1^2+p_1^2}} + \frac{p_1}{\sqrt{m_2^2+p_1^2}} = \frac{p_1(E_1+E_2)}{E_1 E_2}$$

$$g(p_1) = \frac{p_1^2}{4E_1E_2}$$

$$g(p^*)\left|\frac{df}{dp_1}\right|_{p^*} = \frac{|\vec{p}^*|}{4(E_1+E_2)} = \frac{|\vec{p}^*|}{4m_a}$$

$$\Rightarrow \Gamma_{fi} = \frac{p^*}{32\pi^2 m_a^2}\int |M_{fi}|^2 d\Omega = \frac{1}{\tau} \qquad 1\,\Gamma_{fi}$$

$$p^* = \frac{1}{2m_a}\sqrt{[m_a^2-(m_1+m_2)^2][m_a^2-(m_1-m_2)^2]}$$

from $f(p_1)=0$ (energy conservation)

Before integration: $\frac{1}{8\pi^2 m_a}$ After int $\frac{p_f^*(p^*)}{32\pi^2 m_a^2}$

$\Rightarrow$ Integral $\int |M_{fi}|^2\,\delta(E_i-E_1-E_2)\,\delta^3(\vec{p}_1+\vec{p}_2)\,\frac{d^3\vec{p}_1}{2E_1}\cdot\frac{d^3\vec{p}_2}{2E_2} =$

$$= \frac{1}{4E_a}\cdot|\vec{p}_f^*|\int |M_{fi}|^2 d\Omega$$